Exercise 21

Find the limit or show that it does not exist.

$$\lim_{x\to\infty} \frac{(2x^2+1)^2}{(x-1)^2(x^2+x)}$$

Solution

Expand the numerator and denominator. Then multiply the numerator and denominator by the reciprocal of the highest power of x in the denominator.

$$\begin{aligned}
\lim_{x\to\infty} \frac{(2x^2+1)^2}{(x-1)^2(x^2+x)} &= \lim_{x\to\infty} \frac{4x^4+4x^2+1}{(x^2-2x+1)(x^2+x)} \\
&= \lim_{x\to\infty} \frac{4x^4+4x^2+1}{x^4-x^3-x^2+x} \\
&= \lim_{x\to\infty} \frac{4x^4+4x^2+1}{x^4-x^3-x^2+x} \cdot \frac{\frac{1}{x^4}}{\frac{1}{x^4}} \\
&= \lim_{x\to\infty} \frac{(4x^4+4x^2+1)\frac{1}{x^4}}{(x^4-x^3-x^2+x)\frac{1}{x^4}} \\
&= \lim_{x\to\infty} \frac{4+\frac{4}{x^2}+\frac{1}{x^4}}{1-\frac{1}{x}-\frac{1}{x^2}+\frac{1}{x^3}} \\
&= \frac{\lim_{x\to\infty}\left(4+\frac{4}{x^2}+\frac{1}{x^4}\right)}{\lim_{x\to\infty}\left(1-\frac{1}{x}-\frac{1}{x^2}+\frac{1}{x^3}\right)} \\
&= \frac{4+\lim_{x\to\infty}\frac{4}{x^2}+\lim_{x\to\infty}\frac{1}{x^4}}{\lim_{x\to\infty}1-\lim_{x\to\infty}\frac{1}{x}-\lim_{x\to\infty}\frac{1}{x^2}+\lim_{x\to\infty}\frac{1}{x^3}} \\
&= \frac{4+0+0}{1-0-0+0} \\
&= 4
\end{aligned}$$